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DYNAMIC CRACK PROPAGATION IN A LAYER

K. B. BROBERG
Department of Mathematical Physics, University College Dublin, Belfield, Dublin 4, Ireland

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Abstract—Mode I crack propagation in an elastic layer is discussed. Previous investigations have dealt with cases in which the crack edge travels with sub-Rayleigh velocity with respect to the surrounding medium, and with either sub-Rayleigh or supersonic velocity with respect to the layer. In the present paper, the third fundamental case is studied; crack edge travel at sub-Rayleigh velocity with respect to the layer and supersonic velocity with respect to the outer medium. Comparison is made between the three cases, and the theoretical basis for expecting the plane stress Rayleigh wave velocity of the outer medium as an upper limit of the crack velocity in the recent experiments by Washabaugh and Knauss (1994, *Int. J. Fract.* **65**, 97–114) is discussed.

1. INTRODUCTION

Crack propagation in a layer is of interest in connection with earthquakes, structural joints and laminate materials. Since crack edges sometimes reach velocities which may be higher than half the Rayleigh wave velocity, different situations can occur, depending on the relation between crack edge velocity and the sound velocities in the layer and in the surrounding medium. Even though the crack edge velocity usually is sub-Rayleigh with respect to both media, it might be super-Rayleigh and even intersonic or supersonic with respect to one of the media, if these are very dissimilar. However, under mode I loading, the velocity must be sub-Rayleigh with respect to at least one of the media, except in cases when energy can be fed directly to the crack edge, as at crack face loading.

In previous papers, the cases of supersonic (Broberg, 1974, 1977) and sub-Rayleigh (Broberg, 1975) mode I crack propagation with respect to the layer were investigated, assuming sub-Rayleigh velocity with respect to the surrounding medium. In the present paper the crack edge velocity is assumed to be sub-Rayleigh with respect to the layer, and supersonic with respect to the surrounding medium. For simplicity, as in the two previous papers, steady state conditions, obtained by moving crack face loads, are assumed. However, it will be shown that the validity of the solution obtained can be extended to cases of remote loading. This is also possible, at least approximately, for the other two cases, as briefly discussed by Broberg (1977).

2. STATEMENT OF THE PROBLEM

A layer of thickness $2d$ of a linearly elastic material is situated in an infinite solid. The propagation velocities for the solid are c for P-waves and kc for S-waves. For the layer, the propagation velocities are κc for P-waves and $k_1 \kappa c$ for S-waves. It is assumed that $k_1 \kappa > 1$, i.e. the propagation velocity for S-waves in the layer is higher than the propagation velocity of S-waves in the solid. The modulus of rigidity is μ for the solid and μ_1 for the layer. By using μ, μ_1, k and k_1 as elastic constants, the treatment is valid both for plane strain and plane stress.

Due to the symmetry, only half the body needs to be regarded. Thus, introducing a Cartesian coordinate system x, y, the solid $y \leqslant 0$ and the layer $0 \leqslant y \leqslant d$ are regarded. The lower crack surface is situated on $y = d, x < Vt$, where t is time and V is a constant velocity.

A normal stress $\sigma_y = h(x - Vt)$ is acting on the surface $y = d, x < Vt$. $h(x)$ is assumed to be insignificant for $x < -L$, i.e. the "extension" of the load can be described by L. It is assumed that $1 < V/c < k_1\kappa$, i.e. the crack velocity is supersonic with respect to the semi-infinite solid and subsonic with respect to the layer. On the surface $y = d, x > Vt$, the normal displacement $v = 0$. The shear stress $\tau_{xy} = 0$ on the whole surface $y = d$.

The problem consists essentially of finding the normal stress $\sigma_y = f(x - Vt)$ on the surface $y = d, x > Vt$, and the normal displacement $v(x - Vt)$ on the surface $y = d, x < Vt$.

3. TREATMENT OF THE PROBLEM

The problem is treated in essentially the same way as the one in Broberg (1974). Thus, a solution is first sought to the simple boundary value problem of a concentrated force P, acting on $y = d$ and moving with constant velocity V in the positive x direction. In addition to the boundary conditions for $y = d$, there are continuity conditions for normal and shear stresses as well as for the displacements along the interface $y = 0$. It is convenient to introduce a moving coordinate system with dimensionless coordinates,

$$\xi = (x - Vt)/d \tag{1}$$

$$\eta = y/d \tag{2}$$

and the functions

$$v_1(\xi, \eta) = v(x - Vt, y) \tag{3}$$

$$h_0(\xi) = h(x - Vt), \quad \text{for } \xi < 0, \quad \eta = 1. \tag{4}$$

Then the equations of motion are

$$a_1^2 \frac{\partial^2 \Phi}{\partial \xi^2} - \frac{\partial^2 \Phi}{\partial \eta^2} = 0 \tag{5}$$

$$a_2^2 \frac{\partial^2 \Psi}{\partial \xi^2} - \frac{\partial^2 \Psi}{\partial \eta^2} = 0 \tag{6}$$

for the solid and

$$b_1^2 \frac{\partial^2 \Phi_1}{\partial \xi^2} + \frac{\partial^2 \Phi_1}{\partial \eta^2} = 0 \tag{7}$$

$$b_2^2 \frac{\partial^2 \Psi_1}{\partial \xi^2} + \frac{\partial^2 \Psi_1}{\partial \eta^2} = 0 \tag{8}$$

for the layer. Here Φ, Ψ, Φ_1 and Ψ_1 are potential functions such that

$$u_1 = \frac{\partial \Phi}{\partial \xi} + \frac{\partial \Psi}{\partial \eta}, \quad u_1 = \frac{\partial \Phi_1}{\partial \xi} + \frac{\partial \Psi_1}{\partial \eta} \tag{9}$$

$$v_1 = \frac{\partial \Phi}{\partial \eta} - \frac{\partial \Psi}{\partial \xi}, \quad v_1 = \frac{\partial \Phi_1}{\partial \eta} - \frac{\partial \Psi_1}{\partial \xi} \tag{10}$$

where u_1 is the displacement in the ξ direction. Furthermore

$$a_1^2 = \frac{V^2}{c^2} - 1, \qquad a_1 > 0$$

$$a_2^2 = \frac{V^2}{k^2 c^2} - 1, \qquad a_2 > 0$$

$$b_1^2 = 1 - \frac{V^2}{\kappa^2 c^2}, \qquad b_1 > 0$$

$$b_2^2 = 1 - \frac{V^2}{k_1^2 \kappa^2 c^2}, \qquad b_2 > 0.$$

The solution to the simple boundary value problem involves a very time-consuming determination of constants, whereupon one obtains the displacement gradient $\partial v_1/\partial \xi$ on $\eta = 1$ for the concentrated force,

$$\left(\frac{\partial v_1}{\partial \xi}\right)_{\eta=1} = -\frac{Pa_1(1+a_2^2)}{\pi\mu[4a_1 a_2 + (1-a_2^2)^2]} \cdot \lim_{\delta\to 0} \int_0^\infty e^{-\delta\alpha} \frac{N(\alpha)}{D(\alpha)} e^{i\alpha\xi}\, d\alpha \tag{11}$$

where

$$N(\alpha) = 1 + R \tanh(b_1\alpha) - S \tanh(b_2\alpha) + \frac{a_2 b_1}{a_1 b_2} \tanh(b_1\alpha)\tanh(b_2\alpha)$$

$$\begin{aligned} D(\alpha) = {} & \operatorname{sech}(b_1\alpha)\operatorname{sech}(b_2\alpha) + \frac{\mu_1}{\mu} \cdot \frac{A}{b_1(1-b_2^2)} \\ & \times \{[(1+b_2)^2 R + 4b_1 b_2 S] \cdot [\operatorname{sech}(b_1\alpha)\operatorname{sech}(b_2\alpha) - 1] \\ & + \left[4\frac{a_2 b_1^2}{a_1} - (1+b_2^2)^2\right] \tanh(b_1\alpha) + \left[4b_1 b_2 - \frac{a_2 b_1}{a_1 b_2}(1+b_2^2)^2\right] \tanh(b_2\alpha) \\ & + [4b_1 b_2 R + (1+b_2^2)^2 S] \tanh(b_1\alpha)\tanh(b_2\alpha)\} \end{aligned}$$

$$A = -\frac{ia_1(1+a_2^2)}{4a_1 a_2 + (1-a_2^2)^2}$$

$$R = Cb_1\left\{4\frac{\mu_1}{\mu}(1+a_1 a_2) - 4(1-a_2^2+2a_1 a_2) + \frac{\mu}{\mu_1}[4a_1 a_2 + (1-a_2^2)]\right\}$$

$$S = \frac{C}{b_2}\left\{\frac{\mu_1}{\mu}(1+a_1 a_2)(1+b_2^2)^2 - 2(1-a_2^2+2a_1 a_2)(1+b_2^2) + \frac{\mu}{\mu_1}[4a_1 a_2 + (1-a_2^2)]\right\}$$

$$C = \frac{i}{a_1(1+a_2^2)(1-b_2^2)}.$$

Now, by superposition, an expression in the form of an integral is obtained for the displacement gradient $\partial v_1/\partial x$ on $\eta = 1$ resulting from an arbitrarily distributed load $\sigma_y = f_1(\xi)$ moving with velocity V on $\eta = 1$

$$g_1(\xi) = -\int_{-\infty}^{+\infty} f_1(\zeta) \int_0^\infty \frac{N(\alpha)}{D(\alpha)} e^{i\alpha(\xi-\zeta)}\, d\alpha\, d\zeta \tag{12}$$

where

$$g_1(\xi) = \frac{\pi\mu[4a_1a_2+(1-a_2^2)^2]}{a_1(1+a_2^2)}\cdot\left(\frac{\partial v_1}{\partial \xi}\right)_{\eta=1}. \tag{13}$$

The stress $f_1(\xi)$ and the displacement gradient function $g_1(\xi)$ are now written as

$$f_1(\xi) = f_0(\xi)+h_0(\xi), \quad f_0(\xi) = 0 \text{ for } \xi < 0, \quad h_0(\xi) = 0 \text{ for } \xi > 0 \tag{14}$$

$$g_1(\xi) = g_0(\xi), \quad g_0(\xi) = 0 \text{ for } \xi > 0. \tag{15}$$

It is now obvious that eqn (12) contains as unknown quantities the displacement gradient function $g_0(\xi)$ for $\xi < 0$ and the normal stress $f_0(\xi)$ for $\xi > 0$. It is a Wiener–Hopf equation of the first kind. In order to solve the equation, the following Laplace transforms are introduced:

$$F(p) = p\int_0^{+\infty} e^{-p\xi} f_0(\xi)\,d\xi, \quad \mathscr{R}p \geqslant 0 \tag{16}$$

$$H(p) = p\int_{-\infty}^{0} e^{-p\xi} h_0(\xi)\,d\xi, \quad \mathscr{R}p \leqslant 0 \tag{17}$$

$$G(p) = p\int_{-\infty}^{0} e^{-p\xi} g_0(\xi)\,d\xi, \quad \mathscr{R}p \leqslant 0. \tag{18}$$

Since the integral in eqn (12) is a convolution integral, and

$$p\int_{-\infty}^{+\infty} e^{-(p\pm i\alpha)\xi}\,d\xi = \mp 2\pi i\alpha\delta(ip\mp\alpha), \quad \mathscr{R}p = 0 \tag{19}$$

where $\delta(\alpha)$ is Dirac's delta function, one obtains

$$G(p) = -\pi[F(p)+H(p)]K(p), \quad \mathscr{R}p = 0 \tag{20}$$

where

$$K(p) = \frac{N(-ip)}{D(-ip)}. \tag{21}$$

The first step in the Wiener–Hopf technique is to factorize $K(p)$, i.e. to write

$$K(p) = K_+(p)K_-(p), \quad \mathscr{R}p = 0 \tag{22}$$

where $K_+(p)$ is regular in $\mathscr{R}p \geqslant 0$ and $K_-(p)$ is regular in $\mathscr{R}p \leqslant 0$. In order to perform the factorization, some properties of $N(\alpha)$ and $D(\alpha)$ need to be known. Both possess imaginary zeroes only (see the Discussion), and

$$\frac{N(\infty)}{D(\infty)} = -\frac{N(-\infty)}{D(-\infty)} = i\left|\frac{N(\infty)}{D(\infty)}\right| \tag{23}$$

$$\ln\left[\frac{N(\alpha)}{D(\alpha)}\right] \to \ln\left[\frac{N(\pm\infty)}{D(\pm\infty)}\right] \pm i\frac{\pi}{2} \text{ as } \alpha \to \pm\infty. \tag{24}$$

When $\alpha \to 0$,

$$\ln \frac{N(\alpha)}{D(\alpha)} \to iA_0\alpha + B_0\alpha^2, \quad A_0, B_0 \text{ real.} \tag{25}$$

Now, by standard methods, one obtains

$$\ln K_+(p) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\left[\ln\frac{N(q)}{D(q)}\right]' \ln(q+ip)\,dq + \tfrac{1}{2}\ln\frac{N(-\infty)}{D(-\infty)}, \quad \mathscr{R}p \geqslant 0 \tag{26}$$

$$\ln K_-(p) = -\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\left[\ln\frac{N(q)}{D(q)}\right]' \ln(q+ip) + \tfrac{1}{2}\ln\frac{N(-\infty)}{D(-\infty)}, \quad \mathscr{R}p \leqslant 0 \tag{27}$$

where $'$ sign denotes differentiation with respect to q. As $p \to 0$,

$$\ln K_+(p) \to \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\left[\ln\frac{N(q)}{D(q)}\right]' \ln|q|\,dq$$

$$= -\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\left[\ln\frac{\overline{N(q)}}{\overline{D(q)}}\right]' \ln|q|\,dq = \overline{\ln K_+(p)} \tag{28}$$

where a bar denotes complex conjugation. Thus,

$$K_+(p) \to C_0 \text{ as } p \to 0 \tag{29}$$

where C_0 is a real constant. Similarly one obtains

$$K_-(p) \to C_0^{-1} \text{ as } p \to 0. \tag{30}$$

As $p \to \infty$,

$$\ln K_+(p) \to \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\left[\ln\frac{N(q)}{D(q)}\right]'\left(\ln p + i\frac{\pi}{2}\right)dq + \tfrac{1}{2}\ln\frac{N(-\infty)}{D(-\infty)}$$

$$= \frac{\ln p}{2\pi i}\ln\frac{N(\infty)D(-\infty)}{D(\infty)N(-\infty)} + \tfrac{1}{4}\ln\frac{N(\infty)N(-\infty)}{D(\infty)D(-\infty)} \tag{31}$$

and, since

$$\frac{N(\infty)D(-\infty)}{D(\infty)N(-\infty)} = e^{\pi i}, \tag{32}$$

one obtains

$$K_+(p) = \left|\frac{N(\infty)}{D(\infty)}\right|^{1/2} p^{1/2} = C_\infty p^{1/2}. \tag{33}$$

Similarly, as $p \to -\infty$,

$$K_-(p) \to -C_\infty(-p)^{-1/2}. \tag{34}$$

For sub-Rayleigh crack velocity with respect to the layer, the constant C_∞ is found to be

$$C_\infty = \left\{\frac{\mu b_1(1-b_2^2)[4a_1a_2+(1-a_2^2)^2]}{\mu_1 a_1(1+a_2^2)[4b_1b_2-(1+b_2^2)^2]}\right\}^{1/2}. \tag{35}$$

The second step in the Wiener–Hopf technique is to make necessary partitions so that each term is regular in either the left or the right half-plane. Insertion of eqn (22) into eqn (20) gives

$$\pi F(p)K_+(p)+\pi H(p)K_+(p) = -\frac{G(p)}{K_-(p)}, \quad \mathscr{R}p = 0. \tag{36}$$

The second term does not possess the required character and must be partitioned. To this end one observes that

$$H(p) \to p\int_0^\infty h_0(-\xi)\,\mathrm{d}\xi \text{ as } p \to 0 \tag{37}$$

$$H(p) \to -h_0(0) \text{ as } p \to \pm\mathrm{i}\infty. \tag{38}$$

Thus

$$H(p)K_+(p) \to C_0 p\int_0^\infty h_0(-\xi)\,\mathrm{d}\xi \text{ as } p \to 0 \tag{39}$$

$$H(p)K_+(p) \to -C_0 h_0(0) \text{ as } p \to \pm\mathrm{i}\infty \tag{40}$$

and, hence, the partition

$$\frac{\pi}{p}H(p)K_+(p) = L_+(p)+L_-(p), \quad \mathscr{R}p = 0 \tag{41}$$

is obtained by choosing

$$L_+(p) = \frac{\mathrm{i}}{2}\int_{-i\infty}^{+i\infty}\frac{H(q)K_+(q)}{q(q-p)}\,\mathrm{d}q, \quad \mathscr{R}p \geqslant 0 \tag{42}$$

$$L_-(p) = -\frac{\mathrm{i}}{2}\int_{-i\infty}^{+i\infty}\frac{H(q)K_+(q)}{q(q-p)}\,\mathrm{d}q, \quad \mathscr{R}p \leqslant 0. \tag{43}$$

Thus,

$$\pi F(p)K_+(p)+pL_+(p) = -\frac{G(p)}{K_-(p)}-pL_-(p), \quad \mathscr{R}p = 0. \tag{44}$$

Here the left member is regular for $\mathscr{R}p \geqslant 0$ and the right member is regular for $\mathscr{R}p \leqslant 0$. Both members behave algebraically as p tends towards infinity in the respective regions of regularity. Then, by Liouville's theorem, they must equal a polynomial of finite degree. With $K_0, K_1, \ldots, K_n$ as constants, the third step in the Wiener–Hopf technique gives

$$\pi F(p) = -p\frac{L_+(p)}{K_+(p)}+\frac{K_0}{K_+(p)}+\frac{K_1 p}{K_+(p)}\cdots+\frac{K_n p^n}{K_+(p)}, \quad \mathscr{R}p \geqslant 0 \tag{45}$$

$$G(p) = -pL_-(p)K_-(p)-K_0K_-(p)-K_1pK_-(p)\cdots-K_np^nK_-(p), \quad \mathscr{R}p \leqslant 0. \tag{46}$$

The final step in the Wiener–Hopf technique consists of determining the constants K_i. One observes that

$$\pi C_\infty F(p) \to K_0 p^{-1/2} + K_1 p^{1/2} \cdots + K_n p^{n-1/2} \text{ as } p \to \infty \tag{47}$$

$$\pi C_0 F(p) \to K_0 + [K_1 - L_+(0)]p + K_2 p^2 \cdots + K_n p^n \text{ as } p \to 0. \tag{48}$$

The condition that $f_0(\xi)$ must be integrable gives for $p \to \infty$ from eqn (47)

$$K_2 = K_3 \cdots = K_n = 0 \tag{49}$$

and for $p \to 0$ from eqn (48)

$$K_0 = 0 \tag{50}$$

$$K_1 = L_+(0). \tag{51}$$

Hence

$$\pi F(p) = \frac{[L_+(0) - L_+(p)]p}{K_+(p)}, \quad \mathscr{R}p \geqslant 0 \tag{52}$$

$$G(p) = -[L_+(0) + L_-(p)]pK_-(p), \quad \mathscr{R}p \leqslant 0. \tag{53}$$

This is the solution of the Wiener–Hopf equation (20). Inversion of $F(p)$ and $G(p)$ can be made for $p \to \infty$ and $p \to -\infty$, respectively, giving $f_0(\xi)$ and $g_0(\xi)$ for $|\xi| \ll 1$. Since $L_+(p) \to 0$ as $p \to \infty$ and $L_-(p) \to 0$ as $p \to -\infty$, one obtains

$$F(p) \to \frac{L_+(0)p^{1/2}}{\pi C_\infty} \quad \text{as } p \to +\infty \tag{54}$$

$$G(p) \to -L_+(0)C_\infty(-p)^{1/2} \quad \text{as } p \to -\infty. \tag{55}$$

Hence

$$f_0(\xi) \to \frac{2L_+(0)}{\pi C_\infty \sqrt{\pi\xi}} \text{ as } \xi \to +0 \tag{56}$$

$$g_0(\xi) \to \frac{2L_+(0)C_\infty}{\sqrt{-\pi\xi}} \text{ as } \xi \to -0. \tag{57}$$

For later use, the function $L_+(p)$ will be written in a more manageable form, by the substitution $q \to \mathrm{i}q$ in the integral in eqn (42):

$$L_+(p) = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{H(\mathrm{i}q)K_+(\mathrm{i}q)}{q(q+\mathrm{i}p)} \mathrm{d}q$$

$$= \frac{i}{2}\int_0^{\infty} h_0(-\xi) \int_{-\infty}^{+\infty} \frac{\mathrm{e}^{\mathrm{i}q\xi} K_+(\mathrm{i}q)}{q+\mathrm{i}p} \mathrm{d}q\, \mathrm{d}\xi, \quad \mathscr{R}p \geqslant 0. \tag{58}$$

4. THICK LAYER APPROXIMATION, $d/L \gg 1$

For $p \to 0$, one obtains from eqn (58)

$$L_+(0) = \frac{i}{2}\int_0^{\infty} h_0(-\xi)\, C\int_{-\infty}^{+\infty} \frac{e^{iq\xi}\, K_+(iq)}{q}\, dq\, d\xi + \frac{\pi}{2} C_0 \int_0^{\infty} h_0(-\xi)\, d\xi \tag{59}$$

where C on the integral sign denotes the Cauchy principal value.

For a thick layer $d/L \gg 1$, $K_+(iq)$ can be approximated by $C_\infty (iq)^{1/2}$, so that insertion into eqn (59) gives

$$\begin{aligned} L_+(0) &\approx -\frac{i}{2} C_\infty \int_0^{\infty} h_0(-\xi)\, d\xi \int_{-\infty}^{+\infty} \frac{e^{iq\xi}}{(iq)^{1/2}}\, dq\, d\xi + \frac{\pi}{2} C_0 \int_0^{\infty} h_0(-\xi)\, d\xi \\ &= -\frac{1}{\sqrt{2}} C_\infty \int_0^{\infty} \frac{h_0(-\xi)}{\sqrt{\xi}} \int_0^{\infty} \frac{\cos u + \sin u}{\sqrt{u}}\, du\, d\xi + \frac{\pi}{2} C_0 \int_0^{\infty} h_0(-\xi)\, d\xi. \end{aligned} \tag{60}$$

Thus,

$$\begin{aligned} L_+(0) &\approx -\frac{\sqrt{\pi}}{2} C_\infty \int_0^{\infty} \frac{h_0(-\xi)}{\sqrt{\xi}}\, d\xi + \frac{\pi}{2} C_0 \int_0^{\infty} h_0(-\xi)\, d\xi \\ &= -\frac{\sqrt{\pi}}{2} C_\infty \int_0^{\infty} \frac{h_0(-\xi)}{\sqrt{\xi}}\, d\xi \left[1 - \frac{\sqrt{\pi} C_0 \theta}{C_\infty} \cdot \sqrt{\frac{L}{d}}\right] \end{aligned} \tag{61}$$

where $\theta < 1$, since

$$\left| \int_0^{\infty} \frac{h_0(-\xi)}{\sqrt{\xi}}\, d\xi \right| \geqslant \sqrt{\frac{d}{L}} \int_0^{\infty} h_0(-\xi)\, d\xi. \tag{62}$$

Use of eqns (56) and (57) gives, after returning to original coordinates,

$$(\sigma_y)_{y=d} \to -\frac{1}{\pi} \cdot \frac{1}{\sqrt{x - Vt}} \int_0^{\infty} \frac{h(-s)}{\sqrt{s}}\, ds \left[1 - \frac{\sqrt{\pi} C_0 \theta}{C_\infty} \cdot \sqrt{\frac{L}{d}}\right] \text{as } \frac{x - Vt}{d} \to 0 \tag{63}$$

$$\begin{aligned} \left(\frac{\partial v}{\partial x}\right)_{y=d} &\to -\frac{b_1(1 - b_2^2)}{\pi \mu_1 [4b_1 b_2 - (1 + b_2^2)^2]} \cdot \frac{1}{\sqrt{Vt - x}} \\ &\quad \times \int_0^{\infty} \frac{h(s)}{\sqrt{s}}\, ds \cdot \left[1 - \frac{\sqrt{\pi} C_0 \theta}{C_\infty} \sqrt{\frac{L}{d}}\right] \text{as } \frac{Vt - x}{d} \to 0. \end{aligned} \tag{64}$$

Not surprisingly, as $d/L \to \infty$, one obtains the result by Craggs (1960) for a semi-infinite crack in an infinite body of the layer material. Since the energy release rate is proportional to $\sqrt{x - Vt}(\sigma_y)_{y=d} \sqrt{Vt - x}(\partial v/\partial x)_{y=d}$ (Broberg, 1964), it is obtained, approximately, from the homogeneous case by multiplication with the factor

$$1 - \frac{2\sqrt{\pi} C_0 \theta}{C_\infty} \sqrt{\frac{L}{d}}. \tag{65}$$

From eqn (64), it is obvious that $(\partial v/\partial x)_{y=d}$ changes sign when V passes the Rayleigh wave velocity for the layer. This velocity is thus an upper limit for V.

5. THIN LAYER APPROXIMATION, $d/L \ll 1$

Since

$$K_+(\mathrm{i}q) = \frac{N(q)}{K_-(\mathrm{i}q)D(q)}, \quad \mathscr{I}q = 0, \tag{66}$$

one can substitute $K_+(\mathrm{i}q)$ in eqn (58) by the right member of equality (66). $K_-(\mathrm{i}q)$ is regular for $\mathscr{I}q \geqslant 0$, and therefore the integral over q can be calculated by residue calculus. $D(q)$ possesses imaginary zeroes; $q = \mathrm{i}\beta_n$, $n = 1, 2, 3, \ldots, \beta_n < \beta_{n+1}$. In the neighbourhood of a zero,

$$D(q) \approx D'(\mathrm{i}\beta_n)(q - \mathrm{i}\beta_n). \tag{67}$$

Hence,

$$\begin{aligned} L_+(p) &= \pi\mathrm{i} \int_0^\infty h_0(-\xi) \sum_{n=1}^\infty \frac{\mathrm{e}^{-\beta_n \xi} N(\mathrm{i}\beta_n)}{K_-(-\beta_n) D'(\mathrm{i}\beta_n)(p+\beta_n)} \mathrm{d}\xi \\ &= \pi\mathrm{i} \sum_{n=1}^\infty \frac{N(\mathrm{i}\beta_n)}{K_-(-\beta_n) D'(\mathrm{i}\beta_n)(p+\beta_n)} \int_0^\infty h_0(-\xi)\, \mathrm{e}^{-\beta_n \xi}\, \mathrm{d}\xi. \end{aligned} \tag{68}$$

Hence, use of eqns (56) and (57) gives, after returning to original coordinates,

$$\begin{aligned} (\sigma_y)_{y=d} \to -\frac{2}{\sqrt{\pi}} \sqrt{\frac{d}{x-Vt}} &\left\{ \frac{\mu a_1 (1+a_2^2)[4b_1 b_2 - (1+b_2^2)^2]}{\mu_1 [4a_1 a_2 + (1-a_2^2)^2] b_1 (1-b_2^2)} \right\}^{1/2} \\ &\times \sum_{n=1}^\infty \frac{\mathrm{i}N(\mathrm{i}\beta_n)}{K_-(-\beta_n) D'(\mathrm{i}\beta_n)\beta_n} \int_0^\infty h(-\xi d) \mathrm{e}^{-\beta_n \xi}\, \mathrm{d}\xi \text{ as } \frac{x-Vt}{d} \to 0 \end{aligned} \tag{69}$$

$$\begin{aligned} \left(\frac{\partial v}{\partial x}\right)_{y=d} \to -\frac{2}{\sqrt{\pi}\mu} \sqrt{\frac{d}{Vt-x}} &\left\{ \frac{\mu_1 a_1 (1+a_2^2) b_1 (1-b_2^2)}{\mu[4a_1 a_2 + (1-a_2^2)^2][4b_1 b_2 - (1+b_2^2)^2]} \right\}^{1/2} \\ &\times \sum_{n=1}^\infty \frac{\mathrm{i}N(\mathrm{i}\beta_n)}{K_-(-\beta_n) D'(\mathrm{i}\beta_n)\beta_n} \int_0^\infty h(-\xi d)\, \mathrm{e}^{-\beta_n \xi}\, \mathrm{d}\xi \text{ as } \frac{Vt-x}{d} \to 0. \end{aligned} \tag{70}$$

One notices that $D'(\mathrm{i}\beta_n)$ is imaginary. From eqn (69) it is obvious that $(\sigma_y)_{y=d}$ goes to zero when the Rayleigh wave velocity for the layer is approached. The Rayleigh wave velocity is thus an upper limit of V.

The integral in eqns (69) and (70) is convergent for any finite $h(x)$, and one can thus allow L/d to reach infinity. In particular, one can have $h(x) = \text{constant} = -\sigma_y^\infty$ and then, by superposition of the stress $\sigma_y = \sigma_y^\infty$ everywhere, one obtains the case of traction free crack faces and remote loading:

$$\begin{aligned} (\sigma_y)_{y=d} \to \frac{2\sigma_y^\infty}{\sqrt{\pi}} \sqrt{\frac{d}{x-Vt}} &\left\{ \frac{\mu a_1 (1+a_2^2)[4b_1 b_2 - (1+b_2^2)^2]}{\mu_1 [4a_1 a_2 + (1-a_2^2)^2] b_1 (1-b_2^2)} \right\}^{1/2} \\ &\times \sum_{n=1}^\infty \frac{\mathrm{i}N(\mathrm{i}\beta_n)}{K_-(-\beta_n) D'(\mathrm{i}\beta_n)\beta_n^2} \text{ as } \frac{x-Vt}{d} \to 0 \end{aligned} \tag{71}$$

$$\left(\frac{\partial v}{\partial x}\right)_{y=d} \to \frac{2\sigma_y^\infty}{\sqrt{\pi}}\sqrt{\frac{d}{Vt-x}}\left\{\frac{\mu_1 a_1(1+a_2^2)b_1(1-b_2^2)}{\mu[4a_1a_2+(1-a_2^2)^2][4b_1b_2-(1+b_2^2)^2]}\right\}^{1/2}$$
$$\times\sum_{n=1}^{\infty}\frac{\mathrm{i}N(\mathrm{i}\beta_n)}{K_-(-\beta_n)D'(\mathrm{i}\beta_n)\beta_n^2} \text{ as } \frac{Vt-x}{d}\to 0. \quad (72)$$

This result is perhaps somewhat unexpected, since one is used to thinking that a semi-infinite crack cannot be subjected to a finite constant remote load normal to its plane. However, regarding the case with crack face loading, it is obvious that energy from the load is transferred to the crack edge region only through the layer, and it is thus filtered away more and more the further back from the crack edge it originates. Mathematically this is expressed by the fact that the integrals in eqns (69) and (70) contain exponential functions, so that the contribution from "large" values of the integration variable becomes insignificant, i.e., the "tail" of a "long" moving load does not influence the results appreciably. Consequently the results can be carried over to other cases than semi-infinite cracks; they should also hold approximately for an edge crack or a central crack in a plate as soon as approximately constant velocity is obtained.

The energy release rate $\mathrm{d}W/\mathrm{d}S$ is found in the usual way for dynamic cases as $\pi\sigma_y\sqrt{x-Vt}\,\partial v/\partial x\sqrt{Vt-x}$ as $|x|\to Vt$ (Broberg, 1964):

$$\frac{\mathrm{d}W}{\mathrm{d}S}=\frac{4d}{\mu}\cdot\frac{a_1(1+a_2^2)}{4a_1a_2+(1-a_2^2)^2}\left\{\sum_{n=1}^{\infty}\frac{\mathrm{i}N(\mathrm{i}\beta_n)}{K_-(-\beta_n)D'(\mathrm{i}\beta_n)\beta_n}\int_0^\infty h(-\xi d)\,\mathrm{e}^{-\beta_n\xi}\,\mathrm{d}\xi\right\}^2 \quad (73)$$

which for remote loading reduces to

$$\frac{\mathrm{d}W}{\mathrm{d}S}=\frac{4(\sigma_y^\infty)^2 d}{\mu}\cdot\frac{a_1(1+a_2^2)}{4a_1a_2+(1-a_2^2)^2}\left\{\sum_{n=1}^{\infty}\frac{\mathrm{i}N(\mathrm{i}\beta_n)}{K_-(-\beta_n)D'(\mathrm{i}\beta_n)\beta_n^2}\right\}^2. \quad (74)$$

For finite values of L/d one obtains, from eqn (41),

$$L_+(0)+L_-(0)=\lim_{p\to 0}\frac{\pi H(p)K_+(p)}{p}=\pi C_0\int_0^\infty h_0(-\xi)\,\mathrm{d}\xi \quad (75)$$

and hence, from eqn (53),

$$\frac{G(p)}{p}\to -\pi\int_0^\infty h_0(\xi)\mathrm{d}\xi \text{ as } p\to 0 \quad (76)$$

and thus, according to a Tauber theorem,

$$\lim_{\xi\to-\infty}\int_\xi^0 g(\zeta)\mathrm{d}\zeta = -\pi\int_0^\infty h_0(\zeta)\,\mathrm{d}\zeta \quad (77)$$

i.e.

$$v(x)\to\frac{a_1(1+a_2^2)}{\mu[4a_1a_2+(1-a_2^2)^2]}\int_0^\infty h(s)\,\mathrm{d}s \text{ as } x\to-\infty. \quad (78)$$

For the case $L/d=\infty$, or constant remote loading, one can easily obtain the asymptotic behaviour of $v(x)$ as $x\to-\infty$ by assuming $d=0$. The solution is then obtained from the solution $\Phi=\Phi_0(x-a_1y)$, $\Psi=\Psi_0(x-a_2y)$ to the equations of motion (5) and (6) after use of the boundary conditions on $y = 0$, that $\sigma_y = 0$ for $x > 0$, $\sigma_y = -\sigma_y^\infty$ for $x < 0$ and $\tau_{xy} = 0$ for all x. The result for $x < 0$ is

$$v(x) = \frac{\sigma_y^\infty}{\mu} \cdot \frac{a_1(1+a_2^2)}{4a_1a_2+(1-a_2^2)^2} x \tag{79}$$

which therefore is the asymptotic expression for $v(x)$ as $x \to -\infty$ when d is finite.

6. DISCUSSION

For crack propagation to take place at a higher velocity than the velocity of P-waves in the outer medium, the stress intensity factor must overshoot a certain minimum value. The crack will then propagate with a speed, which is determined by this stress intensity factor, and possibly also other factors, cf Broberg (1979), Ravi-Chandar (1982), Ravi-Chandar and Knauss (1984a–d) and Johnson (1992, 1993).

As is evident from eqn (69) or (71), the thinner the layer is, the higher must the load be to produce the minimum stress intensity factor required. If the layer is thin enough then, at remote loading, the normal stress at the layer boundaries will reach the interface strength before the load is high enough to produce the minimum stress intensity factor required. Thus, at remote loading, a material dependent minimum thickness of the layer is required to obtain a crack velocity according to the conditions of the problem studied.

Now, even if the conditions of the problem can be satisfied as regards interface stresses and stress intensity factors, one might question whether the crack will propagate in the middle of the layer or veer towards one side. Questions about directional stability of cracks produced by wedging have been studied by Melin (1991), assuming slow crack growth in the middle of a strip. She showed that the path remains straight if the thickness of the wedge is smaller than a material constant times the stress intensity factor times the square root of the strip thickness. Although the present case is different in several important aspects, a direct transfer of Melin's results indicates that directional stability ought to prevail if the layer is sufficiently thick. But the minimum thickness required could be much smaller than for the strip in Melin's problem, due to the presence of a supporting solid medium on both sides of the layer. Moreover, unlike the case of a slowly moving crack, the layer boundaries can influence the feedback from a disturbance in the crack edge region only through a narrow wedge shaped region with the apex at the crack edge and directed forwards. On the other hand, there is certainly also a velocity effect which, for instance, might lead to branching in the layer.

Results obtained earlier for two other cases than the one studied here show the following expressions for the crack edge vicinity in the thin layer approximations:

Case 1. Crack velocity sub-Rayleigh with respect to both media (Broberg, 1975):

$$(\sigma_y)_{y=d} \approx -\frac{1}{\pi\sqrt{x-Vt}} \left\{\frac{\mu_1 a_1(1-a_2^2)[4b_1b_2-(1+b_2^2)^2]}{\mu[4a_1a_2-(1+a_2^2)^2]b_1(1-b_2^2)}\right\}^{1/2} \times \int_0^\infty \frac{h(-s)}{\sqrt{s}}\,\mathrm{d}s \text{ as } \frac{x-Vt}{d} \to 0 \tag{80}$$

$$\left(\frac{\partial v}{\partial x}\right)_{y=d} \approx -\frac{1}{\pi\sqrt{Vt-x}} \left\{\frac{a_1(1-a_2^2)b_1(1-b_2^2)}{\mu\mu_1[4a_1a_2-(1+a_2^2)^2][4b_1b_2-(1+b_2^2)^2]}\right\}^{1/2} \times \int_0^\infty \frac{h(-s)}{\sqrt{s}}\,\mathrm{d}s \text{ as } \frac{Vt-x}{d} \to 0 \tag{81}$$

where

$$a_1^2 = 1 - \frac{V^2}{c^2}, \qquad a_1 > 0$$

$$a_2^2 = 1 - \frac{V^2}{k^2c^2}, \qquad a_2 > 0$$

$$b_1^2 = 1 - \frac{V^2}{\kappa^2c^2}, \qquad b_1 > 0$$

$$b_2^2 = 1 - \frac{V^2}{k_1^2\kappa^2c^2}, \quad b_2 > 0$$

and other notations in agreement with those previously introduced in Sections 2 and 3.

Case 2. Crack velocity sub-Rayleigh with respect to the outer medium, supersonic with respect to the layer (Broberg, 1974, 1977):

$$(\sigma_y)_{y=d} \approx -\frac{C_3}{\sqrt{\pi d}}\int_0^\infty \frac{h(-s)}{\sqrt{s}}\,\mathrm{d}s \text{ as } \frac{x-Vt}{d} \to 0 \tag{82}$$

$$(v)_{y=d} \approx \frac{a_1(1-a_2^2)}{C_3\sqrt{\pi}\mu[4a_1a_2-(1+a_2^2)^2]}\sqrt{d}\int_0^\infty \frac{h(-s)}{\sqrt{s}}\,\mathrm{d}s \text{ as } \frac{Vt-x}{d} \to 0 \tag{83}$$

where C_3 is a constant given in Broberg (1974, 1977),

$$a_1^2 = 1 - \frac{V^2}{c^2}, \qquad a_1 > 0$$

$$a_2^2 = 1 - \frac{V^2}{k^2c^2}, \qquad a_2 > 0$$

$$b_1^2 = \frac{V^2}{\kappa^2c^2} - 1, \qquad b_1 > 0$$

$$b_2^2 = \frac{V^2}{k_1^2\kappa^2c^2} - 1, \quad b_2 > 0$$

and other notations in agreement with those previously introduced in Sections 2 and 3.

Comparison between the different cases. The case investigated in the present paper will be referred to as case 0. As regards the treatment of the three problems, there are some similarities. Thus, the function $N(\alpha)/D(\alpha)$ in the present work can be obtained from the function with the same notation in Broberg (1974, 1977) after multiplying a_1, a_2, b_1 and b_2 by the imaginary unit i, or from the function with the same notation in Broberg (1975) after multiplying a_1 and a_2 by i. There are unfortunately misprints in the definitions of $N(\alpha)$ and $D(\alpha)$ in the earlier papers. In the 1974 paper, a factor 2 is missing in front of the second term in the expression for S_2, in the 1977 paper, the terms $R_1 \sin b_1\alpha \cos b_2\alpha$ and $-S_2 \cos b_1\alpha \sin b_2\alpha$ are omitted in the expression for $N(\alpha)$, and in the 1975 paper, the second term in the expression for $D(\alpha)$ should be deleted. In the 1974 paper, it is shown that the functions denoted $D(\alpha)$ and $N(\alpha)$ possess real zeroes, only. This property implies that the functions $N(\alpha)$ and $D(\alpha)$ in the present work possess imaginary zeroes only.

One observes that $(\sigma_y)_{y=d}$ is non-singular in case 2 and that the crack edge is completely blunted. This is a consequence of the fact that the energy flow to the crack edge region originates from the moving load on the crack faces, but it cannot reach the edge region directly through the layer, only indirectly through the outer medium towards the layer boundary somewhat in front of the crack edge and then to the edge region through the layer.

The stress intensity factor in case 1 is independent of the layer thickness $2d$, whereas the stress intensity factor in case 0 is proportional to $\sqrt{d}$, i.e. the stress σ_y decays more rapidly with increasing distance from the crack edge the thinner the layer is at the same crack face load $h(x-Vt)$ or, if $L/d=\infty$, at the same remote load σ_y^∞. This is, of course, the expected behaviour. For case 2, the stress in front of the crack edge is proportional to $1/\sqrt{d}$.

For cases 1 and 2, the dependence on the crack face load $h(x-Vt)$ on stresses and displacements near the crack edge is given by the integral

$$\int_0^\infty \frac{h(-s)}{\sqrt{s}}\,\mathrm{d}s \tag{84}$$

whereas for case 0, this dependence is given by the integrals

$$\int_0^\infty h(-\xi d)\,\mathrm{e}^{-\beta_n \xi}\,\mathrm{d}\xi. \tag{85}$$

Since β_1 is of the order of π (but depends on V/c and material parameters), the first of the integrals for case 0 is of the order of $h(0)/\pi$, irrespective of L, if $h(x-Vt)$ is approximately constant, equal to $h(0)$, for $-a<x<0$, where a is of the order of πd. The integrals following the first one are smaller, and their contribution to the near edge field is much smaller than that of the first integral. Thus, the character of the load at distances larger than about two layer thicknesses from the crack edge has no significant influence on the stress–strain field near the crack edge region provided, of course, that the load at these larger distances is not much larger than $h(0)$.

Case 1 may be of interest in connection with a recent experimental investigation by Washabaugh and Knauss (1994). They succeeded in reaching about 90% of the Rayleigh wave velocity for crack growth in a thin layer of weakened material in a PMMA plate. The explanation obviously is that the process region is prevented from lateral growth, which otherwise seems to lead to increasing energy dissipation per unit of crack growth, although the crack velocity may stay constant, as shown experimentally by Ravi-Chandar (1982) and Ravi-Chandar and Knauss (1984a–d), and by numerical simulations by Johnson (1992, 1993). In one simulation, which can be said to be the numerical equivalent to the experiment by Washabaugh and Knauss (1994), Johnson (1993) confined the process region to a thin layer and then, in contrast to the case with a non-confined process region, the crack did not reach a constant terminal velocity, but continued to accelerate as long as the simulation could be carried out. In other simulations with a confined process region, but for mode II, Johnson (1990) obtained crack velocities even in the intersonic region.

Washabaugh and Knauss (1994) were very careful to obtain the same elasto-dynamic properties at the layer as for the virgin material, whereas the strength, measured as the static fracture toughness, was considerably reduced. However, one can expect some, even though perhaps very small, differences as regards the elasto-dynamic properties. Such a difference would not influence the energy flow to the crack edge region appreciably, cf eqns (80) and (81), but the stress intensity factor would reach zero before the Rayleigh wave velocity for the virgin material is reached, if the layer Rayleigh wave velocity is lower. It should be noticed, though, that the relevant layer Rayleigh velocity should be the one at plane strain, if the layer is much thinner than the plate thickness, as is the case in the experiments by Washabaugh and Knauss, whereas the relevant Rayleigh wave velocity for the virgin material should be the one at plane stress. If, on the other hand, the layer has higher relevant Rayleigh wave velocity than the virgin material, the Rayleigh wave velocity for the virgin material is an upper limit, since eqn (81) signals interpenetration of the crack faces otherwise.

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